JOURNAL OF APPROXIMATION THEORY 37, 119–124 (1983)

Approximate Calculation of Green's Functions

G. ADOMIAN

Center for Applied Mathematics
University of Georgia, Athens, Georgia 30602, U.S.A.

Communicated by Oved Shisha

Received April 13, 1981

The Green's function for a differential equation can be determined by an easily computable series by decomposition of the differential operator into one whose inverse is known or found with little effort and a second component—with no smallness restrictions—whose effects can be determined.

A Green's function which may be difficult to determine in particular cases can be determined using an easily computable series by decomposition of a differential operator L, which is sufficiently difficult to merit approximation, into an operator L_1 whose inverse is known, or found with little effort, and an operator L_2, with no smallness restrictions, whose contribution to the total inverse L^{-1} can be found in series form.

Consider therefore a differential equation $Ly = x(t)$, where L is a linear deterministic ordinary differential operator of the form $L = \sum_{\nu=0}^{n} a_\nu(t)\, d^\nu/dt^\nu$, where a_n is nonvanishing on the interval of interest.

Decompose L into $L_1 + L_2$, where L_1 is sufficiently simple that determination of its Green's function is trivial. Then if L_2 is zero, we have simply $y(t) = \int_0^t l(t, \tau)\, x(\tau)\, d\tau$, where $l(t, \tau)$ is the Green's function for the L_1 operator. If L is a second-order differential operator we may have $L_1 = d^2/dt^2$ and L_2 will be the remaining terms of L, say, $\alpha(t)\, d/dt + \beta(t)$. More generally, $L = \sum_{\nu=0}^{n} a_\nu(t)\, d^\nu/dt^\nu$ and we might take $L_1 = d^n/dt^n$ and $L_2 = \sum_{\nu=0}^{n-1} a_\nu(t)\, d^\nu/dt^\nu$. We have

$$Ly = (L_1 + L_2)\, y = x(t), \tag{1}$$

$$L_1 y = x(t) - L_2 y,$$

$$y = L_1^{-1} x - L_1^{-1} L_2 y. \tag{2}$$

Now assume a decomposition $y = \sum_{i=0}^{\infty} y_i$ and assuming initial conditions are zero, or equivalently, ignoring the homogeneous solution, we identify y_0 as $L_1^{-1} x$ and write

$$y = L_1^{-1} x - L_1^{-1} L_2 (y_0 + y_1 + \cdots) \tag{3}$$

0021-9045/83/020119-06$03.00/0

from which we can determine the y_i—each being determinable in terms of the preceding y_{i-1}. (If the initial conditions are nonzero, they must be included in y_0 as will be discussed shortly.) Thus,

$$y = L_1^{-1}x - L_1^{-1}L_2 y_0 - L_1^{-1}L_2 y_1 - \cdots$$

or

$$y = L_1^{-1}x - L_1^{-1}L_2L_1^{-1}x + L_1^{-1}L_2L_1^{-1}L_2L_1^{-1}x - \cdots. \tag{4}$$

We have

$$y = \sum_{i=0}^{\infty} (-1)^i (L_1^{-1}L_2)^i L_1^{-1}x; \tag{5}$$

thus

$$\begin{aligned}
y_0 &= L_1^{-1}x, \\
y_1 &= -L_1^{-1}L_2L_1^{-1}x, \\
y_2 &= L_1^{-1}L_2L_1^{-1}L_2L_1^{-1}x, \\
&\vdots \\
y_i &= (-1)^i (L_1^{-1}L_2)^i L_1^{-1}x.
\end{aligned} \tag{6}$$

Hence the inverse of the differential operator L is given by

$$L^{-1} = \sum_{i=0}^{\infty} (-1)^i (L_1^{-1}L_2)^i L_1^{-1}. \tag{7}$$

Equation (2) written out explicitly in terms of the Green's function for L_1^{-1} is

$$y = \int_0^t l(t, \tau)\, x(\tau)\, d\tau - \int_0^t l(t, \tau)\, L_2 [y(\tau)]\, d\tau \tag{8}$$

$$= \int_0^t l(t, \tau)\, x(\tau)\, d\tau - \int_0^t L_2^{\dagger} [l(t, \tau)]\, y(\tau)\, d\tau \tag{9}$$

$$= \int_0^t l(t, \tau)\, x(\tau)\, d\tau - \int_0^t \sum_{i=0}^{\infty} (-1)^i \frac{d^i}{d\tau^i} [l(t, \tau)\, a_i(\tau)]\, d\tau. \tag{10}$$

If $L_1 = a_n d^n/dt^n$ and $L_2 = \sum_{i=0}^{n-1} a_\nu(t)\, d^i/dt^i$,

$$y(t) = \int_0^t l(t, \tau)\, x(\tau)\, d\tau - \int_0^t \sum_{i=0}^{n-1} (-1)^i \frac{d^i}{d\tau^i} [l(t, \tau)]\, y(\tau)\, d\tau$$

or

$$y(t) = \int_0^t l(t, \tau)\, x(\tau)\, d\tau - \int_0^t k(t, \tau)\, y(\tau)\, d\tau, \tag{11}$$

where $k(t, \tau) = \sum_{i=0}^{n-1} (-1)^i (d^i/d\tau^i)[l(t, \tau)\, a_i(\tau)]$.

Let's choose $i = 0$; i.e., we work with the case $L = L_1 + \alpha(t)$, i.e., $L_2 = \alpha(t)$. Now

$$y(t) = L_1^{-1}x - L_1^{-1}\alpha(t)\, L_1^{-1}x + L_1^{-1}\alpha L_1^{-1}\alpha L_1^{-1}x - \cdots ; \tag{12}$$

i.e.,

$$y_0 = \int_0^t l(t, \tau)\, x(\tau)\, d\tau,$$

$$y_1 = -\int_0^t l(t, \tau)\, a(\tau) \int_0^\tau l(\tau, \gamma)\, x(\gamma)\, d\gamma\, d\tau,$$

$$y_2 = \int_0^t \int_0^\tau \int_0^\gamma l(t, \tau)\, l(\tau, \gamma)\, l(\gamma, \sigma)\, a(\tau)\, a(\gamma)\, x(\sigma)\, d\gamma\, d\tau\, d\sigma,$$

etc.,

or equivalently

$$y(t) = \int_0^t l(t, \tau)\, x(\tau)\, d\tau - \int_0^t k(t, \tau)\, y(\tau)\, d\tau, \tag{13}$$

where $k(t, \tau) = l(t, \tau)\, a(\tau)$. Hence

$$y(t) = \int_0^t l(t, \tau)\, x(\tau)\, d\tau - \int_0^t k(t, \tau)\, y_0(\tau)\, d\tau + \int_0^t k(t, \tau)\, y_1(\tau)\, d\tau + \cdots .$$

If we let $F(t) = L_1^{-1}x = \int_0^t l(t, \tau)\, x(\tau)\, d\tau$, we can write

$$\begin{aligned} y(t) = F(t) &- \int_0^t k(t, \tau)\, F(\tau)\, d\tau \\ &+ \int_0^t d\tau \int_0^\tau d\gamma\, k(t, \tau)\, k(\tau, \gamma)\, F(\gamma) \\ &- \int_0^t d\tau \int_0^\tau d\gamma \int_0^\gamma d\sigma\, k(t, \tau)\, k(\tau, \gamma)\, k(\gamma, \sigma)\, F(\sigma) \\ &+ \cdots . \end{aligned} \tag{14}$$

If L_1 has constant coefficients, $l(t, \tau) = l(t - \tau)$. For simplicity and clarity, let us now consider the example $L = L_1 + \alpha$ with α a constant and $L_1^{-1} = \int dt$ and the Green's function $l = 1$.

Let us now inquire into our stated objective of determining the Green's function $G(t, \tau)$ for L. G satisfies $LG(t, \tau) = \delta(t - \tau)$ or

$$(L_1 + L_2)G = \delta(t - \tau). \tag{15}$$

Thus G can be found from the preceding equations by replacing x by the δ function. For the last example $L = d/dt + \alpha$,

$$(d/dt + \alpha)\, G(t, \tau) = \delta(t - \tau).$$

If we write $G = G_0 + G_1 + \cdots$, we have immediately (using $l = 1$, $L_2 = \alpha$, $y = G$, $x = \delta(t - \tau)$)

$$G_0 = \int_\tau^t \delta(t - \tau)\, d\tau = 1 \qquad (t > \tau).$$

Remembering $k(t, \tau) = l(t - \tau)\alpha = \alpha$,

$$\begin{aligned} G_1 &= \alpha \int_\tau^t d\tau = \alpha(t - \tau), \\ G_2 &= \int_0^t d\tau \int_0^\tau d\gamma\, \alpha^2 = \alpha^2(t - \tau)/2, \\ &\vdots \end{aligned} \tag{16}$$

Consequently

$$G = 1 + \alpha(t - \tau) + \alpha^2(t - \tau)/2 + \cdots, \tag{17}$$

an approximation to

$$G = e^{-\alpha(t - \tau)}. \tag{18}$$

Physically this equation could model a particle of mass m moving as a result of a force $f(t)$ in a resisting medium:

$$m\, dv/dt + Rv = f(t)$$

or

$$(L + \alpha)v = f(t)/m,$$

where $L = d/dt$ and $\alpha = R/m$. We have

$$(L + \alpha)\, G(t, \tau) = \delta(t - \tau)/m.$$

Now

$$G_0 = L_1^{-1}\frac{\delta(t-\tau)}{m} = \frac{1}{m},$$

$$G_1 = -(\alpha/m)\int_0^t\int_0^\tau \delta(\gamma-\tau)\,d\gamma\,d\tau = -(\alpha/m)(t-\tau),\dots .$$

Thus

$$G = \frac{1}{m}\left[1 - \frac{R}{m}(t-\tau) + \cdots\right]$$

$$\simeq \frac{1}{m}e^{-(R/m)(t-\tau)}, \qquad t > \tau.$$

Thus, we use L_1^{-1} as a first approximation and find the total response function G as a series in which L_2 need not be a perturbation on L_1.

For an example of a second-order differential equation, let $L_1 = d^2/dt^2$ and $L_2 = \alpha d/dt + \beta(t)$. Then $y = L_1^{-1}x - L_1^{-1}L_2L_1^{-1}x + L_1^{-1}L_2L_1^{-1}L_2L_1^{-1}x - \cdots$; hence $G(t,\tau)$ satisfies

$$[d^2/dt^2 + \alpha\, d/dt + \beta]\, G(t,\tau) = \delta(t-\tau).$$

The Green's function $l(t,\tau)$ is now $(t-\tau)$, where $t > \tau$, and $G(t,\tau)$ is again determinable as a series as before. If $G(0,\tau) = G'(0,\tau) = 0$, $G(t,\tau)$ is easily found from the iterative series $G(t,\tau) = L_1^{-1}\delta(t-\tau) - L_1^{-1}L_2G(t,\tau)$ since $G_0 = L_1^{-1}\delta(t-\tau)$ and $G(t,\tau) = G_0(t,\tau) + G_1(t,\tau) + \cdots$.

Initial conditions. If initial conditions are not zero and the equation is second order, the first term for y is not simply $y_0 = L_1^{-1}x$ but $y(0) + ty'(0) + L_1^{-1}x$. This is best seen by writing $L_1y + L_2y = x$ as $L_1y = x - L_2y$ and operating with L_1^{-1} from the left to obtain $L_1^{-1}L_1y = L_1^{-1}x - L_1^{-1}L_2y$. The left hand side involves a double integration of a second derivative resulting in $y(t) - y(0) - ty'(0)$. For nth order equations $y_0 = L^{-1}x + \sum_{\nu=0}^{n-1}(t^\nu/\nu!)\,y^{(\nu)}(0)$.

Convergence. If $l(t,\tau)$ is bounded in the interval of interest, α is bounded, and x is bounded, their bounds can be taken outside the integrals. The remaining n-fold integrals yield an $n!$ in the denominator assuring convergence.

References

1. G. Adomian, "Stochastic Systems," Academic Press, New York, 1983.
2. G. Adomian, Stochastic systems analysis, *in* "Applied Stochastic Processes" (G. Adomian, Ed.), Academic Press, New York, 1980.